Topology Vol. 2, pp. 367–377. Pergamon Press, 1964. Printed in Great Britain

SECONDARY COHOMOLOGY OPERATIONS ON THE THOM CLASS

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(*Received* 26 *June* 1963)

§1. INTRODUCTION

LET $\mu = (E, p, B)$ be an n-plane bundle over a CW-complex B. The Thom complex T_μ is formed by taking the associated unit disk bundle and identifying to a point the bounding unit sphere bundle. The space T_μ is $(n-1)$-connected and if μ is orientable then $H^n(T_\mu; Z) = Z$ generated by U_μ, the Thom class. Associated with the zero cross section of μ is an embedding $i : B \to T_\mu$. Using this embedding we can form the product of cochains of B with a cochain in the class U_μ. This representation gives an isomorphism $H^i(T_\mu) \simeq H^{i-n}(B)$. Using this representation Thom proved $Sq^i U_\mu = U_\mu \cdot w_i(\mu)$ where $w_i(\mu)$ is the ith Stiefel–Whitney class of μ. Our purpose is to investigate secondary cohomology operations which are defined when some w_i are zero.

These results have applications to differential topology and some are given in §4. Most of these applications are either theorems or strengthened versions of theorems announced in [7]. It is Theorem (2.2.1), which we did not have when [7] was written, that enables us to improve the theorems of that paper. Two of the results are as follows. Let N^n be a compact orientable differentiable manifold. Let $\bar{w}_i$ be the normal Stiefel–Whitney classes.

THEOREM (4.2.2). *Suppose $n \not\equiv 3 \bmod 8$ and $\bar{w}_2 = 0$ and $n \geq 9$. If N^n is immersible in R^{2n-4}, then $\bar{w}_4 \cdot \bar{w}_{n-4} = 0$.*

For example (4.2.3.) the quarternion projection space of real dimension $4 \cdot 2^j, j > 1$, does not immerse in $R^{8 \cdot 2^j - 4}$ even though it embeds in $R^{8 \cdot 2^j - 3}$.

The next result is a strengthened form of (5.6) of [7].

COROLLARY (4.2.6). *If $n \equiv 1 \bmod 4$, $n > 5$, $\bar{w}_{n-2} = 0$, then N^n is immersible in R^{2n-3} iff $\bar{w}_2 \cdot \bar{w}_{n-3} \in Sq^1 H^{n-2}(N^n; Z_2)$.*

§2. THE SECONDARY OPERATIONS

(2.1). All the secondary operations with which we will be concerned are based on the following two Adem relations;

$$Sq^2 Sq^{k+1} = \binom{k}{2} Sq^{k+3} \tag{2.1.1}$$

$$Sq^4 Sq^{k+1} + Sq^{k+3} Sq^2 = \binom{k}{4} Sq^{k+5}. \tag{2.1.2}$$

† Both authors were partially supported by the U.S. Army Research Office (Durham) and the second-named author is an Alfred P. Sloan Fellow.

These are relations in the Steenrod algebra A_2 modulo the left ideal generated by Sq^1, considered as acting on $H^*(X; Z)$ with values in $H^*(X; Z_2)$. The universal example for each cohomology operation as acting on a class of dimension n is given by a fiber space E_n over $K(Z, n)$ with fiber F, a product of Eilenberg MacLane spaces, and appropriately chosen k-invariants (see [1]). Consider a particular example. If $k = 2 \bmod 4$ then (2.1.1) becomes $Sq^2 Sq^{k+1} + Sq^1 Sq^{k+2} = 0$. We can consider this as

$$Sq^2(\beta_2 Sq^k) + Sq^1 Sq^{k+2} = 0 \tag{2.1.3}$$

where β_2 is the Bockstein coboundary associated with $0 \to Z \to Z \to Z_2 \to 0$. The fiber $F = K_1(Z, n + k) \times K_2(Z_2, n + k + 1)$ and the k-invariants are $\beta_2 Sq^k$ and Sq^{k+2} respectively. Consider the exact sequence

$$H^*(K(Z, n)) \to H^*(E_n) \xrightarrow{i^*} H^*(F) \xrightarrow{\tau} H^*(K(Z, n))$$

which holds up to $2n + k$ by Theorem IX, (14.1) of [4]. Let f_1 and f_2 be images in $H^*(F)$ of the characteristic classes of K_1 and K_2 respectively. Then $\tau f_1 = \beta_2 Sq^k$ and $\tau f_2 = Sq^{k+2}$ and $\tau(Sq^2 f_1 + Sq^1 f_2) = 0$. Hence there is a class v_n^k in $H^*(E_n; Z_2)$ such that $i^* v_n^k = Sq^2 f_1 + Sq^1 f_2$. Note that v_n^k is not unique but determined only modulo $j^* H^*(K(Z, n))$, that is, modulo a primary operation. Let α be the characteristic class of $K(Z, n)$. Letting $u_n = j^*\alpha$, we call (E_n, u_n, v_n^k) the universal example for an operation ϕ_k based on (2.1.3). The space E_n is determined by giving the fiber, the relation and the k-invariants. We have the following table which defines operations ϕ_k and ϕ_k' based on relations (2.1.1). The general description of the operation depends on k mod 4. All operations are given as being defined on a class of dimension n.

TABLE (2.1.4)

Symbol		F	k-invariant	Remark
ϕ_k	$k = 0 \bmod 4$	$K_1(Z, n + k)$	$\beta_2 Sq^k$	
	$k = 1 \bmod 4$	$K_1(Z_2, n + k)$	Sq^{k+1}	—
	$k = 2 \bmod 4$	$K_1(Z, n + k) \times k) \times K_2(Z_2, n + k + 1)$	$\beta_2 Sq^k$, Sq^{k+2}	
	$k = 3 \bmod 4$	$K_1(Z_2, n + k)$	Sq^{k+1}	defined only if $n < k + 2$
ϕ_k'	$k = 0 \bmod 4$	$K_1(Z_2, n + k - 1)$	Sq^k	the relation used involves $(Sq^2 Sq^1) Sq^k$
	$k = 2 \bmod 4$	$K_1(Z_2, n + k - 1) \times K_2(Z_2, n + k + 1)$	Sq^k, Sq^{k+2}	

The next table defines operations based on (2.1.2). The dependence on k is more complicated and we let $k = 4s + j$.

TABLE (2.1.5)

Symbol	k	F	k-invariant
ψ_k	$s, j = 0 \bmod 2$	$K(Z, n + k) \times K(Z_2, n + 1)$	$\beta_2 Sq^k$, Sq^2
	$s = 0 \bmod 2$, $j = 1 \bmod 2$	$K(Z_2, n + k) \times K(Z_2, n + 1)$	Sq^{k+1}, Sq^2
	$s = 1 \bmod 2$, $j = 0 \bmod 4$	$K(Z, n + k) \times K(Z_2, n + 1) \times K(Z_2, n + k + 4)$	$\beta_2 Sq^k$, Sq^2, Sq^{k+4}
	$s = 1 \bmod 2$, $j = 2 \bmod 4$	$K(Z, n + k) \times K(Z_2, n + 1) \times K(Z_2, n + k + 2)$	$\beta_2 Sq^k$, Sq^2, Sq^{k+2} (*Note:* $Sq^3 Sq^{k+2} = Sq^{k+5}$)
	$s = 1 \bmod 2$, $j = 3 \bmod 4$	not defined	

(2.2). Each operation defined above has values on classes for which the k-invariant vanishes, and unless otherwise noted, is defined on all dimensions. In particular for every k, ϕ_k and ϕ_k' are defined on the fundamental class of $K(Z, k)$ and $K(Z, k-1)$ respectively. If we let Y be the fiber space over $K(Z, k)$ having $K(Z_2, k+1)$ as fiber and Sq^2 as k-invariant then ψ_k is defined on the fundamental class of Y. We now prove the following theorem.

THEOREM (2.2.1). *In the notation above, letting α be the fundamental class of $K(Z, k)$, $K(Z, k-1)$ or Y, we can choose operations ϕ_k, ϕ_k', and ψ_k such that*

$$\alpha \cup Sq^2\alpha \in \phi_k(\alpha), \qquad 0 \in \phi_k'(\alpha), \qquad \alpha \cup Sq^4\alpha \in \psi_k(\alpha).$$

Notice that if $k = 2$, ϕ_k is closely related to the operation ψ of [1] (4.2), and Theorem (4.4.1) of [1] can be readily deduced from this result.

The proof of Theorem (2.2.1) will use some of the results of Barcus and Meyer [2]. Let SK be the suspension of $K(Z, k)$ and let $s : H^{j+1}(SX) \to H^j(X)$ be the suspension isomorphism. In [2], the Postnikov tower of SK is determined up to dimension $3k$.

Let

$$\begin{tikzcd} & X^1 \arrow[d, "p_1"] & K(Z, 2k+1) \arrow[l] \\ SK \arrow[ur, dashed, "h_2"] \arrow[r, "h_1"'] & K(Z, k+1) & \end{tikzcd} \tag{2.2.2}$$

be the first portion of the non-trivial part. The first non-trivial k-invariant is $b \cup b \in H^{2k+2}(K(Z, k+1); Z)$ where b is the fundamental class of $K(Z, k+1)$ and $h_1^* b = b'$ where $sb' = \alpha$. Consider the sequence

$$H^*(K(Z, k+1)) \xrightarrow{p^*} H^*(X^1) \xrightarrow{i^*} H^*(K(Z, 2k+1)) \to H^*(K(Z, k+1)). \tag{2.2.3}$$

Since $Sq^2(b \cup b) = 0$, there is a class $v \in H^{2k+3}(X^1; Z_2)$ (not uniquely determined) such that $i^* v = Sq^2 f$ where f is the characteristic class of $K(Z, 2k+1)$. Because of (4.1), part II of [2] v is not part of any higher k-invariant and so $h_2^* v \neq 0$ for any choice of v. Let $Sq^I b$ be a basis for $H^{2k+3}(K(Z, k+1); Z_2)$. It is easy to see that h_1^* is a monomorphism, and the class γ such that $s\gamma = \alpha \cup Sq^2\alpha$ is linearly independent (in the vector space sense over the field Z_2) of $h_1^*(H^{2k+3}(K(Z, k+1); Z_2))$. Therefore $h_2^* p_1^*$ is a monomorphism and $h_2^* v$ must be linearly independent of im $h_2^* p_1^*$ and so v can be chosen so that $h_2^* v = \gamma$.

Now observe that if $k = 0 \bmod 4$, $(X^1, p_1^* b, v)$ is just the universal example for a possible choice of ϕ_k and $\gamma \in \phi_k(b')$ implies $s\gamma \in \phi_k(sb')$ or $\alpha \cup Sq^2\alpha \in \phi_k(\alpha)$.

If $k = 2 \bmod 4$ then the space of the universal example for ϕ_k acting on $SK(Z, k)$ is just $E_{k+1} = X^1 \times K_2$ where $K_2 = K(Z_2, 2k+2)$. Let f_2 be the image of the characteristic class of K_2 in $H^*(E_{k+1})$; let v be identified with the image of v; and let $p_2 : E_{k+1} \to K(Z, k+1)$ be the composite fibering. Then the universal example for ϕ_k is $(E_{k+1}, p_2^* b, v + Sq^1 f)$ Let $\lambda : SK \to K_2$ be a trivial map. Then $h = (h_2, \lambda) : SK \to E_{k+1}$ is such that $ph = h_1$ and $h^*(v + Sq^1 f) = h_2^* v$ So again by choosing ϕ_k to have v as its universal example we have $\gamma \in \phi_k(b')$ or $\alpha \cup Sq^2\alpha \in \phi_k(\alpha)$.

If $k = 1 \bmod 2$, then the universal example for ϕ_k is given by a fibering $E_{k+1} \to K(Z, k+1)$ having as fiber $K(Z_2, 2k+1)$ and $(b \cup b) \bmod 2 = Sq^{k+1}b$ as k-invariant. Let $*$ be the base point of $K(Z, k+1)$. Then the map $p_1 : (X^1, F) \to (K(Z, k+1), *)$ can be lifted to a map $g: (X^1, F) \to (E_{k+1}, K(Z_2, 2k+1))$. Consider the sequences

$$\begin{tikzcd} H^*(K(Z, k+1)) \arrow[r, "j_1^*"] \arrow[dr, "j_2^*"'] & H^*(E_{k+1}) \arrow[r, "i_1^*"] \arrow[d, "g^*"] & H^*(K(Z_2, 2k+1)) \arrow[r, "\tau_1"] \arrow[d, "g_1^*"] & H^*(K(Z, k+1) \\ & H^*(X^1) \arrow[r, "i_2^*"] & H^*(K(Z, 2k+1)) \arrow[ur, "\tau_2"'] & \end{tikzcd}$$

Let f_3 be the characteristic class of $K(Z_2, 2k+1)$ and f_4 be the mod 2 version of the characteristic class of $K(Z, 2k+1)$. Since $(b \cup b) \bmod 2 = Sq^{k+1}b$, $g_1^* f_3 = f_4$ Since $Sq^2 Sq^{k+1} b = 0$, there is a class v' such that $i_1^* v' = Sq^2 f_3$ Now $g^* v' - v = j_2^* a$ for some $a \in H^*((KZ, k+1))$ and so v' can be chosen so that $g^* v' = v$. Again we have $\gamma = h_2^* g^* v'$ and so $\gamma \in \phi_k(b')$.

To complete the proof of the theorem for the part dealing with ϕ_k we require that the universal example (E_n, u_n, v_n^k) be chosen so that $s^{n-k+1} v_n^k = v$ (or v'). Then ϕ_k is a stable secondary operation which satisfies the theorem. Note that if $k = 0 \bmod 4$, then ϕ_k is not unique but that $\phi_k + Sq^{k+2}$ will also satisfy the theorem if ϕ_k does.

We now look at ψ_k. The arguments are similar to the above and so we only sketch them. Let ρ: $Y_1 \to K(Z, k+1)$ be the fibering having $Sq^2 b$ as k-invariant and $K(Z_2, k+2)$ as fiber. The diagram (2.2.2) induces, using (1.1) of Part II of [2],

$$\begin{tikzcd} & X^1 \arrow[dd, "\bar{p_1}"] \\ SY \arrow[ur, "\bar{h_2}"] \arrow[dr, "\bar{h_1}"'] & \\ & Y_1 \end{tikzcd}$$

Using $Sq^4(b \cup b) = 0$, as we used $Sq^2(b \cup b) = 0$ before, everything goes through with only minor modifications.

The statement concerning ϕ_k' follows from this general lemma, whose proof is standard and is left to the reader.

LEMMA (2.2.4). *If ϕ_k' is defined on a class a, then ϕ_k is also defined and for a suitable choice of both ϕ_k and ϕ_k', $\phi_k(a) \supset \phi_k' a$*

Now consider $K(Z, k-1)$. The characteristic class α is such that ϕ_k' and therefore ϕ_k, is defined. By the first part of the theorem we see that $0 \in \phi_k(\alpha)$ and, because the indeterminacy $Sq^2 H^{2k-1}(K(Z, k-1))$ consists only of stable classes, we can choose ϕ_k' so that $0 \in \phi_k'(\alpha)$.

DEFINITION (2.2.4). *Let v be the class in $H^{2k+3}(X^1; Z_2)$ defined just below equation (2.2.3). Let $\bar{\phi}_k$ be the secondary cohomology operation having $(x^1, p_1^* b, v)$ as its universal example.*

This operation is defined only on classes of dimension $k+1$ or less and the proof of (2.2.1) gives

THEOREM (2.2.5). *If α is the fundamental class of $K(Z, k)$, then $\alpha \cup Sq^2\alpha \in \tilde{\phi}_k(\alpha)$.*

(2.3). For some applications we will need the following 'Cartan formula':

THEOREM (2.3.1). *If $k = 0 \bmod 2$ and $a \in H^n(X; Z)$ is such that ϕ_k is defined, then there exists a cohomology operation $\bar{\phi}_k$ such that for any $s \in H^1(X; Z_4)$, not of order 2, $\bar{\phi}_k(s \cdot a) \cap s \cdot \phi_k(a)$ is not empty.*

Proof. We will do $k = 0 \bmod 4$, the other case being similar. Let (E_n, u_n, v_n^k) be the universal example for ϕ_k. Let $(F_n, \bar{u}_n, \bar{v}_n^k)$ be the universal example based on $Sq^2Sq^{k+1} = 0$, a relation in the cohomology of $H^*(K(Z_4, n))$. We have

$$\begin{array}{ccc} & & F_n \leftarrow K_4(Z_4, n+k) \\ & & \downarrow \\ E_{n-1} \times K_2(Z_4, 1) & \xrightarrow{\mu'} & K_1(Z_4, n). \end{array}$$

Let the fiber of E_{n-1} be $K_5(Z, n+k)$ and define $\mu' : (E_{n-1} \times K_2, K_5 \times K_2) \to (K_1, *)$ by $\mu'^*\alpha_1 = u_{n-1} \otimes \alpha_2$.

The map μ' can be lifted to a map $\mu : (E_{n-1} \times K_2, K_5 \times K_2) \to (F_n, K_4)$.

Consider the sequences

$$\begin{array}{ccccc} H^*(E_{n-1} \times K_2) & \xrightarrow{i_2^*} & H^*(K_5 \times K_2) & \xrightarrow{\delta_2^*} & H^*(E_{n-1} \times K_2, K_5 \times K_2) \\ \uparrow \mu^* & & \uparrow \mu^* & & \uparrow \mu^* \\ H^*(F_n) & \longrightarrow & H^*(K_4) & \xrightarrow{\delta_1^*} & H^*(F_n, K_4). \end{array}$$

Using Z_2 for coefficients throughout we see

$$\mu^*\delta_1^*\alpha_4 = \mu^*Sq^{k+1}\bar{u}_n = Sq^{k+1}(u_{n-1} \otimes \alpha_2) = (Sq^{k+1}u_{n-1}) \otimes \alpha_2.$$

Therefore $\delta_2^*\mu^*\alpha_4 = \delta^*(\alpha_5 \otimes \alpha_2)$ Hence. $\mu^*i_1^*\bar{v}_n^k = Sq^2(\alpha_5 \otimes \alpha_2) = (Sq^2\alpha_5) \otimes \alpha_2 = i_2^*v_{n-1}^k \otimes \alpha_2$ Hence for a suitable choice of $\bar{v}_n^k$, $\mu^*\bar{v}_n^k = v_{n-1}^k \otimes \alpha_2$ This implies the theorem since for any s and a we have a map $f: X \to E_n \times K_2$ such that $f^*(u_{n-1}) = a$ and $f^*(\alpha_2) = s$. Hence $(f^*v_{n-1}^k) \cup s \in s \cup \phi_k a$ but $(f^*v_{n-1}^k) \cup s = (\mu f)^*\bar{v}_n^k \in \bar{\phi}_k(a \cup s)$.

§3. SECONDARY COHOMOLOGY OPERATIONS IN THE THOM COMPLEX

(3.1). Let G_n be the classifying spaces for $SO(n)$ bundles and let γ_n be the universal n-plane bundle. Let w_i be the ith Stiefel–Whitney class. We will identify w_n and the Euler class and if $i = 2j$ we identify w_{i+1} and $\beta_2 w_i$. If v is any $SO(n)$ bundle over a base space B we have a classifying map $f: B \to G_n$. We write $w_i(v) = f^*(w_i)$. Let $\rho: E_{k,m}^1 \to G_{k+m}$, $n = k + m$, be the fibering defined in [6]. We recall the construction here. The fiber is $\hat{X}^1 = \prod_{j=1}^4 K_j(J_j, k_j)$ with k-invariant k_j^1 defined by

$$k_j^1 = \begin{cases} w_{k+j} & j = 1; j = 2, k = 0 \bmod 2; j = 3, k = 1 \mathit{ mod } 4; j = 4, k = 0 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

and $k_j = k + j - 1$ if $k_j^1 \neq 0$ and -1 otherwise. (Let $K(G, -1)$ be a point.) The group $J_j = Z$ if $m = j$ or $k = 0 \bmod 2$ and $j = 1$, and $J_j = Z_2$ otherwise.

Let $\lambda : G_k \to G_n$ be the usual embedding. Let λ^- and $\tilde{G}_1$ be defined by

$$\begin{array}{ccc} \tilde{G}_1 & \xrightarrow{\lambda^-} & E^1_{k,m} \\ \downarrow & & \downarrow \rho \\ G_k & \xrightarrow[\lambda]{} & G_{k+m} \end{array}$$

The following table defines the symbols we will use for the various Thom complexes we need. Let $n = k + m$.

bundle	γ_n	$\lambda^{-1}\gamma_n$	$\rho^{-1}\gamma_n$	$(\rho\lambda^-)^{-1}\gamma_n$
Thom complex	T_n	$T_{k,m}$	$T^1_{k,m}$	$\tilde{T}^1_{k,m}$
Thom class	U_n	$U_{k,m}$	$U^1_{k,m}$	$\tilde{U}^1_{k,m}$

(3.2). We recall

LEMMA (3.2.1). [Thom]. *Let $\mu = (E, p, B)$ be an orientable n-plane bundle. T_μ is $(n-1)$-connected and if we let i be the embedding of B in T_μ given by the zero cross section then for each cocycle a in B the map $a \to U_\mu \cup a$ induces an isomorphism of $H^i(B)$ with $H^{i+n}(T_\mu)$.*

We also need this easy result:

LEMMA (3.2.2). *There is a canonical map from $f: ST_{k,m} \to T_{k,m+1}$ such that f^* is an isomorphism.*

We now prove

THEOREM (3.2.3). *In the above notation, for all m and for all k where ϕ_k and ϕ_k' are defined, $U_{k,m} \cdot w_2 \cdot w_k \in \phi_k(U_{k,m})$ and $0 \in \phi_k'(U_{k-1,m})$.*

Proof. Because of (3.2.2) it will be sufficient to prove the theorem when $m = 0$. In this case (2.2.1) applies giving $U_k \cdot Sq^2 U_k \in \phi_k U_k$. Now $U_k \cdot Sq^2 U_k = U_k \cdot U_k \cdot w_2 = (Sq^k U_k) \cdot w_2 = U_k \cdot w_k \cdot w_2$. Finally $0 \in \phi_k'(U_{k-1})$ by (2.2.1) and this completes the proof.

(3.3). We recall briefly some of the results of [6]. Suppose ν is an $SO(n)$ bundle over B. Let ν^m be the associated $V_{n,m}$ bundle where $V_{n,m}$ is the space of orthonormal m frames in R^n. Let $n - m = k$. If ν^m is to have a cross section over the $k + 1$ skeleton then $w_{k+1}(\nu) = 0$. But it is clear that if ν^m is to have a cross section over all of B then $w_i(\nu) = 0$ for all $i \geq k + 1$. Suppose B has cohomology dimension $k + 4$. Consider the following diagram:

$$\begin{array}{ccc} & & E^1_{k,m} \\ & \overset{g^1}{\nearrow} & \downarrow \rho \\ B & \xrightarrow[g]{} & G_{k+m} \end{array}$$

where g is the classifying map of ν. If ν has a cross section then g can be lifted to a map g^1. One of the key results of [6] was to identify certain cohomology classes in $H^*(E^1_{k,m})$, k^2_j, such that in order for ν to have a cross section we must find a lifting g^1 such that $g^{1*}(k^2_j) = 0$. Further, this much was sufficient to imply ν has a cross section over the $k + 2$ skeleton and, if $m \geq 3$ and $k = 2$ or $3 \bmod 4$, a cross section over the $k + 3$ skeleton. It also gives necessary conditions for a cross section over the $k + 4$ skeleton.

Finally set

(3.3.1). $k^2_j(\nu) = \{b \in H^*(B; Z_2)$ there exists g^1 such that $g^{1*}k^2_j = b\}$. In the above notatio n we have

THEOREM (3.3.2).

$$U^1_{k,1}\cdot(k^2_1 + w_2\cdot w_k) \in \tilde{\phi}_k U^1_{k,1}$$
$$U^1_{k,m}\cdot(k^2_1 + w_2\cdot w_k) \in \phi_k U^1_{k,m},\ m > 1 \text{ if } \phi_k \text{ is defined (see 2.1.4)}$$
$$U^1_{k,m}\cdot(k^2_2) \in \phi'_{k-1} U^1_{k,m},\ m > 1.$$

Proof. It is clear that the proof will depend on the exact definition of k^2_j which is a function of k and m and is given in [6]. The proof for each case is very similar so we will recall the definition of k^2_1 for $m \geq 2$ and $k = 0 \bmod 4$ and carry through all of the details for this case only.

Suppose $k = 0 \bmod 4$ and $m \geq 2$. Consider the sequence

$$H^*(E^1_{k,m}) \xrightarrow{\lambda^{-*}} H^*(\tilde{G}_1) \xrightarrow{\delta^*} H^*(G_n, G_k)$$

which is exact up to dimension $2k$. Since $\lambda^*(w_{k+j}) = 0$, $\tilde{G}_1$ is topologically $G_k \times X^1_k$. Let x_i be the image of the characteristic class of K_i (of X^1_k) in $\tilde{G}_1$. By the construction we have $\delta^* x_i = w_{k+i}$ if x_i is not zero. Since $(Sq^2 + w_2\cdot)w_{k+1} = 0$ for $k = 0 \bmod 4$, $\delta^*(Sq^2 + w_2\cdot]x_1 = 0$ and so there is a class k^2_1 in $H^{k+2}(E^1_{k,m})$ such that $\lambda^{-*}k^2_1 = (Sq^2 + w_2\cdot)x_1$ and actually this equation defines k^2_1 uniquely.

Now let (E_n, u_n, v^k_n) be the universal example for ϕ_k and let $\omega : E_m \to K(Z, n)$ be the fibering associated with it. The fiber is $K(Z, n + k)$ and we let f represent its fundamental class. The following diagram serves to define all the new spaces we need.

$$\begin{tikzcd}
T_{k,m} \times K_1 \simeq T^-_{k,m} \arrow[r, "\lambda_1"] \arrow[d] & T^-_n \arrow[r, "\mu_1"] \arrow[d] & E_n \arrow[d] \\
T_{k,m} \arrow[r, "\lambda_1"'] & T_n \arrow[r, "\mu"'] & K_2(Z, n).
\end{tikzcd}$$

where $\mu^*(\alpha) = U_n$ and α is the characteristic class of $K(Z, n)$.

Now the natural map $\rho\colon (T^1_{k,m}, \tilde{T}^1_{k,m}) \to (T_n, T_{k,m})$ can be lifted to a map $\bar{\rho}\colon (T^1_{k,m}, \tilde{T}^1_{k,m}) \to (T^-_n, T^-_{k,m})$ since the only obstruction is $Sq^{k+1}U_{k,m} = U_{k,m}\cdot w_{k+1}$ and this is zero. Consider the sequences

$$\begin{tikzcd}
H^*(E_n) \arrow[r, "i_1^*"] \arrow[dd, "\mu_1^*"] & H^*(K_1) \arrow[r, "\delta_1^*"] \arrow[d, "\mu_1^*"'] & H^*(E_n, K_1) \arrow[d, "\mu_1^*"'] \\
 & H^*(K_1) \arrow[r, "\delta_2^*"] & H^*(T^-_n, K_1) \\
H^*(T^-_n) \arrow[ur, "i_2^*"] \arrow[dr, "i_3^*"] \arrow[dd, "\bar{\rho}^*"'] & & \\
 & H^*(T^-_{k,m}) \arrow[r, "\delta_3^*"] \arrow[uu, "\lambda_2^*"'] \arrow[d, "\bar{\rho}^*"] & H^*(T^-_n, T^-_{k,m}) \arrow[uu, "\lambda_2^*"'] \arrow[d, "\bar{\rho}^*"] \\
H^*(T^1_{k,m}) \arrow[r, "i_4^*"] & H^*(\tilde{T}^1_{k,m}) \arrow[r, "\delta_4^*"] & H^*(T^1_{k,m}, \tilde{T}^1_{k,m})
\end{tikzcd}$$

where the remaining maps are defined in a natural fashion. If $\sigma\colon T_{k,m} \to T^{-}_{k,m}$ is an embedding then $g = \mu_1 i_3 \sigma : T_{k,m} \to E_n$ is a map such that $g^* u_n = U_{k,m}$ and, using (3.2.3), we find a choice of σ such that $g^* v^k_m = U_{k,m} \cdot w_2 \cdot w_k$. Hence $i_3^* \mu_1^* v_n^k = (U_{k,m} \cdot w_2 \cdot w_k) \otimes 1 + 1 \otimes (Sq^2 f')$ where f' is chosen such that $\sigma^* f' = 0$ and $\lambda_2^* f' = f$. Now $\delta_4^*(\tilde{U}^1_{k,m} \cdot x_1) = U^1_{k,m} \cdot w_{k+1} = \bar{\rho}^* \delta_3^* f'$. Hence $\bar{\rho}^* f' = \tilde{U}^1_{k,m} \cdot x_1 + i_4^* U^1_{k,m} \cdot \beta$.
Therefore

$$\begin{aligned}\bar{\rho}^* i_3^* \mu_1^* v_n^k &= \tilde{U}^1_{k,m} \cdot w_2 \cdot w_k + Sq^2(\tilde{U}^1_{k,m} \cdot x_1) + i_4^* Sq^2(\tilde{U}_{k,m} \cdot \beta') \\ &= \tilde{U}^1_{k,m} \cdot (w_2 \cdot w_k + i_4^* k_1^2 + (Sq^2 + w_2 \cdot) i_4^* \beta').\end{aligned}$$

This implies the theorem for this case.

For each remaining case a very similar argument works and the details are left to the reader.

(3.4). Let $\pi : K_n \to G_n$ be the fibering with fiber $K(Z_2, 1)$ and w_2 as k-invariant. Clearly K_n is the universal example for fiber bundles having zero 2nd Stiefel–Whitney classes. In general we will identify $\pi^* w_i$ with w_i. (One should note that π is not an isomorphism, for example $\pi^* w_5 = 0 = \pi^* w_9$.) The following diagram defines $M^1_{k,m}$

$$\begin{array}{ccc} M^1_{k,m} & \xrightarrow{\pi_1} & E^1_{k,m} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ K_n & \xrightarrow[\pi]{} & G_n. \end{array}$$

The class k_3^2 defined in [6] has a non-zero projection in $H^{k+4}(M^1_{k,m})$ and we identify $\pi_1^* k_3^2$ with k_3^2. We also have the corresponding spaces

$$\begin{array}{ccc} K_k \times X^1_k \simeq \tilde{M}_{k,m} & \xrightarrow{\lambda_1} & M^1_{k,m} \\ \downarrow & & \downarrow{\scriptstyle \rho} \\ K_k & \xrightarrow[\lambda]{} & K_{k+m}. \end{array}$$

We make the following notational definitions:

(3.4.1)	bundle	$\lambda^{-1}\gamma_{k+m}$	$\rho^{-1}\gamma_{k+m}$	$(\rho\lambda_1)^{-1}\gamma_{k+m}$
	space	K_k	$M^1_{k,m}$	$\tilde{M}_{k,m}$
	Thom complex	$S_{k,m}$	$S^1_{k,m}$	$\tilde{S}^1_{k,m}$
	Thom class	$V_{k,m}$	$V^1_{k,m}$	$\tilde{V}^1_{k,m}$

We have natural maps $\lambda\colon S_k \to S_n$, $\lambda_1 : \tilde{S}^1_{k,n} \to S^1_{k,n}$.

THEOREM (3.4.2). *In the above notation*

$$V_{k,m} \cdot w_4 \cdot w_k \in \psi_k(V_{k,m})$$

for all m and all k such that ψ_k is defined.

First observe that there is a natural map of $SS_{k,m} \to S_{k,m+1}$ which induces isomorphisms of cohomology in all dimensions and so it is sufficient to prove the theorem for $m = 0$. In this case (2.2.1) applies and gives the result as in (3.2.3).

The idea of the proof of (3.3.2), making use of the definitions in [6], yields the following

THEOREM (3.4.3). *If $m \geq 5$ and $k \not\equiv 7$ mod 8, then*

$$V^1_{k,m} \cdot (k_3{}^2 + w_4 \cdot w_k) \in \psi_k(V^1_{k,m}).$$

The details are left to the reader.

(3.5). We now collect all these results together to give

THEOREM (3.5.1). *Let $v = (E, p, B)$ be an n-plane bundle such that $w_{k+j}(v) = 0$ $1 \leq j \leq 4$. Then*

$$U_v \cdot (k_1^2(v) + w_2 \cdot w_k) = \tilde{\phi}_k U_v \qquad \text{if } n = k+1$$

$$U_v \cdot (k_1^2(v) + w_2 \cdot w_k) = \phi_k U_v \qquad n > k+1 \text{ and } \phi_k \text{ is defined}$$

$$U_v \cdot (k_2^2(v)) = \phi'_{k-1} U_v \qquad n > k+1, \quad k = 1 \text{ mod } 2.$$

If in addition $w_2(v) = 0$, then

$$U_v \cdot (k_3^2(v) + w_4 \cdot w_k) = \psi_k U_v \qquad \text{if } m \geq 5 \text{ and } k \not\equiv 7 \text{ mod } 8.$$

Remark. $k_j^2(v)$ is defined in (3.3.1).

Proof. After (3.3.2) and (3.4.3), we need only check that the indeterminacy is correct. But noting that $Sq^2(U_v \cdot \beta) = U_v \cdot (Sq^2 + w_2 \cdot)\beta$ and similarly for $Sq^2 Sq^1$ the first part is clear. For the second part the indeterminacy of ψ_k includes $Sq^{k+3} H^{n+1}(T_v; Z_2)$ and this must be zero since w_{k+2} and w_{k+3} are zero for v. Hence the only indeterminacy of ψ_k which remains is $Sq^4 H^{n+k}(T_v)$ and sometimes $Sq^1 H^{n+k+3}(T_v)$, $Sq^2 H^{n+k+2}(T_v)$ or$Sq^3 H^{n+k+1}(T_v)$. But $Sq^4(U_v \cdot \beta) = U_v \cdot (Sq^4 + w_4 \cdot)\beta$ and $Sq^i U_v \cdot \beta = U_v \cdot Sq^i \beta$ if $i < 4$. Now a check with the table (4.3.3) of [6] shows that these indeterminacies also agree.

Remark. A relation involving ψ_k for values of $m < 5$ is also valid but since it represents only a part of the fourth obstruction and since the result depends on k and m in a complicated way we state only this much. Our statement applies to all stable bundles. —

§4. APPLICATIONS TO DIFFERENTIAL TOPOLOGY

(4.1). In this section N^n will be a closed orientable differentiable manifold. Let $\bar{w}_i$ be the Stiefel–Whitney classes of the normal bundle Hirsch [3] proved that if η_k is the normal bundle to an embedding of N^n in R^{n+k} and if η_k^r has a cross section, then N^n immerses in R^{n+k-r}. If we take $k = n+1$, then the normal bundle is stable and N^n immerses in R^{2n+1-r} if η_{n+1}^r has a cross section. We let η be this normal bundle.

The most useful device for evaluating cross sections is the following result:

(4.1.1). *In the Thom complex T_{η_k}, the class in dimension $n+k$ is spherical. In particular any cohomology operation which takes its values in dimension $n+k$ is zero and has zero indeterminacy.*

This result is well known and follows from the existence of Massey's sub-algebra [8], for example.

(4.2). Our first application is a second proof of (5.4) of [7] (cf. (5.1.1) of [6]).

THEOREM (4.2.1). *Let N^n be a compact orientable n-manifold ($n > 4$). If $n = 2^j$ then N^n immerses in R^{2n-2}* iff $\bar{w}_2 \cdot \bar{w}_{n-2} = 0$. *If $n \neq 2^j$, then N^n immerses in R^{2n-2} if $\bar{w}_2 . \bar{w}_{n-2} =$ 0. If $n \neq 2^j$, then N^n immerses in R^{2n-2}*

Proof. Since N^n is orientable, $\bar{w}_{n-1} = 0$. If $n = 1$ mod 4, then since $\pi_{n-1}(V_{n+1,3}) = 0$, the bundle η^3 always has a cross-section. For all other n we apply ϕ_{n-2} to the Thom class using (3.5.1). Then by (4.1.1) we have

$$k_1^2(\eta) + \bar{w}_2 \cdot \bar{w}_{n-2} = 0.$$

By the results of [6] k_1^2 is the secondary and final obstruction to a cross-section of η_{n+1}^3 and therefore $k_1^2(\eta) = 0$ iff $\bar{w}_2 \cdot \bar{w}_{n-2} = 0$. By a result of Massey and Peterson [9], $\bar{w}_{n-2} = 0$ except possibly if $n = 2^j + i$ and $i = 0$ or 1.

The following is a new result:

THEOREM (4.2.2.) *Suppose $n \neq 3$ mod 8, $n \geq 9$ and $\bar{w}_2 = 0$. If N^n is immersible in R^{2n-4}, then $\bar{w}_4 \cdot \bar{w}_{n-4} = 0$.*

Proof. Let η be the normal bundle to an embedding of N^n in R^{2n+1}. The hypotheses imply that η^5 has a cross-section and so in particular $\bar{w}_{n-j} = 0, 0 \leq j \leq 3$. Hence we can apply ψ_{n-4} to U_η and by (3.5.1) and (4.1.1) we have

$$k_3^2(\eta) = \bar{w}_4 \cdot \bar{w}_{n-4}.$$

(By (4.1.1) the indeterminacy is zero.) But the cross-section of η^5 also implies $k_3^2(\eta) = 0$.

COROLLARY (4.2.3). *If Q_n is quaternionic space of real dimension $4n$ and if $n = 2^j$, $j > 1$, then Q_n does not immerse in R^{8n-4} but it embeds in R^{8n-3}.*

Proof. The dual Stiefel-Whitney classes of Q_n are readily computed by Wu's formula and for $n = 2^j$ satisfy $\bar{w}_4 \cdot \bar{w}_{4n-4} \neq 0$. On the other hand James [5] gives the embedding required.

THEOREM (4.2.4). *If $n \equiv 2$ mod 4 and $n > 4$, then N^n is immersible in R^{2n-3}.*

Proof. By [6] the only remaining obstruction is $k_2^2(\eta)$. Applying ϕ_k' to U using (4.1.1) and (3.5.1) we have the theorem.

THEOREM (4.2.5). *Let $A = Sq^1 H^{n-2}(N^n; Z_2)$, $n > 5$ and $\bar{w}_{n-2} = 0$. If $n = 1$ mod 4, then the second obstruction to finding a cross-section in η^4 is $\bar{w}_2 \cdot \bar{w}_{n-3} + A$. If $n \equiv 3$ mod 4, then the second obstruction differs from $\bar{w}_2 \cdot \bar{w}_{n-3}$ by a class in A.*

Proof. By a result of Massey [8], $(Sq^2 + w_2 \cdot)H^{n-3}(N^n; Z) = 0$ and so if $n = 3$ mod 4, the second obstruction is a single class. Let v be the single class and suppose $\bar{w}_2 \cdot \bar{w}_{n-3} + v \notin A$. Let s be a class in $H^1(N; Z_2)$ such that $s \cdot (v + \bar{w}_2 \cdot \bar{w}_{n-3})$ is non-zero and $Sq^1(s) = 0$. This can be done by Poincaré duality with integer coefficients. Let B and B' be the unit sphere bundle and $n + 1$ disk bundle respectively associated with η and let a be a class such that $q^* \delta^* a = U$ where δ^* is defined by

$$H^*(B)' \longrightarrow H^*(B) \xrightarrow{\delta^*} H^*(B', B) \overset{q^*}{\simeq} H^*(T).$$

We can consider $H^*(N)$ as a subalgebra of $H^*(B)$ and we identify classes this way. Then $\bar{\phi}_k(a \cdot s)$ is defined and, since $q^* \delta^*$ is a natural isomorphism, according to (4.1.1), it must be zero. But, according to (2.3.1) it must be $s \cdot \phi_k(a) = a \cdot (v + \bar{w}_2 \cdot \bar{w}_{n-3}) \cdot s \neq 0$. Hence $v + \bar{w}_2 \cdot \bar{w}_{n-3} \in A$.

If $n = 1$ mod 4, then the second obstruction is a coset of A. Let v be any class in this coset and suppose $v + \bar{w}_2 \cdot \bar{w}_{n-3} \notin A$. Then $\bar{\phi}_k(a \cdot s) = 0$ by (4.1.1) while $s \cdot \phi_k a = a \cdot s \cdot (v + \bar{w}_2 \cdot \bar{w}_{n-3} + A)$ which is a single non-zero class. This gives the theorem.

Since if $n = 1$ mod 4, the third obstruction to a cross-section of η^4 is always zero we have

COROLLARY (4.2.6). *If* $n \equiv 1$ mod 4, $n > 5$, $\bar{w}_{n-2} = 0$, *then* N^n *is immersible in* R^{2n-3} *iff* $\bar{w}_2 \cdot \bar{w}_{n-3} \in Sq^1 H^{n-2}(N^n; Z_2)$.

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